

BLOCH'S CONJECTURE AND VALENCES OF CORRESPONDENCES FOR K3 SURFACES

CLAUDIO PEDRINI

ABSTRACT. Bloch's conjecture for a surface X over an algebraically closed field k states that every homologically trivial correspondence Γ acts as 0 on the Albanese kernel $T(X_\Omega)$, where Ω is a universal domain containing k . Here we prove that, for a complex K3 surface X , Bloch's conjecture is equivalent to the existence of a valence for every correspondence. We also give applications of this result to the case of a correspondence associated to an automorphism of finite order and to the existence of constant cycle curves on X . Finally we show that Franchetta's conjecture, as stated by K.O'Grady, holds true for the family of polarized K3 surfaces of genus g , if $3 \leq g \leq 6$.

1. INTRODUCTION

The existence of a suitable filtration for the Chow ring of every smooth projective variety over a field k , as conjectured by Bloch and Beilinson (see [MNP, Ch.7]), has many important consequences both in arithmetic and in geometry. Apart from the trivial case of curves and some other particular cases, this conjecture is still wide open. Jannsen [Jan, 2.1] has shown that the existence of a Bloch -Beilinson filtration for every smooth projective variety is equivalent to Murre's conjectures on the existence of a suitable Chow-Künneth decomposition of the motives in $\mathcal{M}_{rat}(k)$.

Here $\mathcal{M}_{rat}(k)$ denotes the (covariant) category of Chow motives with rational coefficients over the field k , which is a \mathbf{Q} -linear, pseudoabelian, tensor category. If X is a smooth irreducible, projective variety of dimension d the motive of X is (X, Δ_X) . If X and Y are smooth projective varieties, then $\text{Hom}_{\mathcal{M}_{rat}}(h(X), h(Y)) = A^d(X \times Y)$, where $A^*(X \times Y) = CH^*(X \times Y) \otimes \mathbf{Q}$. If $f : X \rightarrow Y$ is a morphism, then the correspondence $\Gamma_f \in A^2(X \times Y)$ is an element of $\text{Hom}_{\mathcal{M}_{rat}}(h(X), h(Y))$. A smooth irreducible projective surface (over any field k) has a refined Chow-Künneth decomposition $\sum_{0 \leq i \leq 4} h_i(X)$ where $h_0(X) \simeq \mathbf{1}$,

Date: October 21, 2015.

$h_4(X) \simeq \mathbf{L}^2$, $h_2(X) = h_2^{alg}(X) + t_2(X)$ and $t_2(X) = (X, \pi_2^{tr})$, see [KMP, 2.2]. Here

$$\pi_2^{alg}(X) = \sum_{1 \leq h \leq \rho} \frac{[D_h \times D_h]}{D_h^2}.$$

with $\{D_h\}$ an orthogonal basis of $NS(X) \otimes \mathbf{Q}$ and $\rho = \rho(X)$ the rank of $NS(X)$. Therefore $(X, \pi_2^{alg}) \simeq \mathbf{L}^{\oplus \rho}$. We also have

$$\begin{aligned} A_i(t_2(X)) &= \pi_2^{tr} A_i(X) = 0 \text{ for } i \neq 0 ; \\ A_0(t_2(X)) &= \text{Hom}_{\mathcal{M}}(\mathbf{1}, t_2(X)) \simeq T(X) \end{aligned}$$

where $T(X)$ is the Albanese kernel. The transcendental motive $t_2(X)$ is independent of the construction of the Chow-Künneth decomposition, it is functorial for the action of correspondences and a birational invariant for X . The motive $h(X)$ is finite dimensional, in the sense of Kimura iff $t_2(X)$ is evenly finite dimensional, i.e. $\wedge^n t_2(X) = 0$, for some $n > 0$, see [KMP, Thm. 7.6.12]. A consequence of the Bloch-Beilinson's conjectures, or equivalently of Murre's conjectures, is the following Bloch's conjecture for surfaces.

Conjecture 1.1. *Let X be a smooth projective surface over a field k of characteristic 0. Let $\Gamma \in A^2(X \times X)_{hom}$ be a homologically trivial correspondence. Then Γ acts as 0 on the Albanese kernel $T(X_\Omega)$, where Ω is a universal domain containing k .*

If $p_g(X) = q(X) = 0$ then Bloch's conjecture implies $A_0(X)_0 = 0$. In this form Bloch's conjecture is known to hold for all complex surfaces not of general type and for some classes of surfaces of general type, see [PW 2]. From the results of Bloch and Srinivas in [B-S], it follows that Bloch's conjecture for a complex surface X , with $p_g(X) = q(X) = 0$, is equivalent to the class Δ_X of the diagonal in $A^2(X \times X)$ having valence $v(\Delta_X) = 0$, see [PW 1, 4.1]. Here, if Γ is a correspondence in $A^2(X \times X)$, Γ has a valence $v(\Gamma)$ iff $\Gamma + v(\Gamma)\Delta_X \in \mathcal{I}(X)$, where $\mathcal{I}(X) \subset A^2(X \times X)$ denotes the ideal of degenerate correspondences, see [Fu, 16.1.5]. If $p \in A^2(X \times X)$ is a projector which has a valence, then $v(p)$ is either 0 or -1. Since Δ_X has always valence -1 if $v(\Delta_X) = 0$ then Δ_X has two different valences. Also, if $v(\Delta_X) = 0$, then rational, algebraic, homological and numerical equivalence coincide in $A^*(X)$, see [Vois 4]. On the other hand, if $p_g(X) \neq 0$, then the valence of a correspondence is either unique or undefined.

Let X be a smooth projective surface with $p_g(X) \neq 0$. A natural question to ask is to find a relation between conjecture 1.1 and the existence of a valence for every correspondence in $A^2(X \times X)$.

In Sect. 2 we prove that a complex K3 surface X satisfies Bloch's conjecture 1.1 iff every correspondence in $A^2(X \times X)$ has a valence (see Theorem 2.9).

In Sect.3 we consider the case of a K3 surface X over \mathbf{C} with a finite group G of automorphisms of order n . D.Huybrechts in [Huy 1] proved that a symplectic automorphism of finite order acts as the identity on $A_0(X)$. Here (see Theorem 3.3) we show that, for any finite group of automorphisms G on a K3 surface, the projector $p = (1/n) \sum_{g \in G} \Gamma_g$ has a valence. More precisely $v(p) = -1$ if G consists of symplectic automorphisms and $v(p) = 0$ if the automorphisms are non-symplectic. Then we apply these results to compute the virtual number of coincidence for the correspondence p (Corollary 3.5).

Huybrechts in [Huy 2] has introduced the notion of a constant cycle curve on surface X . In Sect. 4 we prove a motivic characterization of a constant cycle curve (Theorem 4.6) which implies a 1-1 correspondence between constant cycle curves on a K3 surface X , fixed by a symplectic automorphism g , and constant cycle curves on the K3 surface Y obtained as a minimal desingularization of X/g , see Corollary 4.7. We also prove (Theorem 4.9) that every curve of fixed points in a correspondence which has a valence $\neq -1$ is a constant cycle curve. Examples of constant cycle curves on K3 surfaces obtained in this ways are given in Ex 4.8 and 4.10.

Finally, in Section 5, we consider the case of a smooth projective family $f : \mathcal{X} \rightarrow S$ of K3 surfaces over a smooth base S , a case which is related to the so called generalized Franchetta's conjecture (see [O'Gr, 5.3]), where \mathcal{X} is the universal family of K3 surfaces with a polarization of degree $2g - 2$ and trivial automorphism group. In Theorem 5.6 we prove that Franchetta's conjecture is equivalent to $A_0(X_\eta) \simeq \mathbf{Q}$, where X_η is the generic fibre of f . Using this result we show (see Corollary 5.7) that the conjecture holds true if $g = 3, 4, 5$, in which cases the projective model in \mathbf{P}^g of a general polarized K3 surface of genus g is a complete intersection projective of $g - 2$ hyper surfaces and for $g = 6$ (see Corollary 5.9).

We thank C.Weibel and K.O'Grady for useful comments on a preliminary version of this paper.

2. VALENCES OF CORRESPONDENCES ON K3 SURFACES

If X is a K3 surface over \mathbf{C} , then it has a refined Chow-Künneth decomposition $h(X) = \sum_{0 \leq i \leq 4} h_i(X)$ with $h_1(X) = h_3(X) = 0$, because X has no odd cohomology. Also $h_2(X) = h_2^{alg}(X) + t_2(X)$, with

$\pi_2 = \pi_2^{alg} + \pi_2^{tr}$, $t_2(X) = (X, \pi_2^{tr})$ and $h_2^{alg} = (X, \pi_2^{alg}) \simeq \mathbf{L}^{\oplus \rho(X)}$. Therefore

$$(2.1) \quad h(X) = \mathbf{1} \oplus \mathbf{L}^{\oplus \rho} \oplus t_2(X) \oplus \mathbf{L}^2$$

where $\rho = \rho(X)$ is the rank of $NS(X)_{\mathbf{Q}} = (Pic X)_{\mathbf{Q}}$, so that $1 \leq \rho(X) \leq 20$. We also have

$$H^i(t_2(X)) = 0 \text{ for } i \neq 2; \quad H^2(t_2(X)) = \pi_2^{tr} H^2(X, \mathbf{Q}) = H_{tr}^2(X, \mathbf{Q}),$$

$$\dim H^2(X, \mathbf{Q}) = b_2(X) = 22; \quad \dim H_{tr}^2(X, \mathbf{Q}) = 22 - \rho(X),$$

$$A_i(t_2(X)) = \pi_2^{tr} A_i(X) = 0 \text{ for } i \neq 0; \quad A_0(t_2(X)) = A_0(X)_0$$

Here $T(X) = A_0(X)_0$ because $q(X) = 0$. We also have

$$\mathrm{Hom}_{\mathcal{M}_{rat}}(\mathbf{1}, t_2(X)) \simeq A_0(X)_0$$

If X and Y are surfaces there is a map, see [KMP, 7.4]

$$\Psi_{X,Y} : A^2(X \times Y) \rightarrow \mathrm{Hom}_{\mathcal{M}_{rat}}(t_2(X), t_2(Y))$$

sending Γ to $\pi_2^{tr}(Y) \circ \Gamma \circ \pi_2^{tr}(X)$, satisfying the following functorial relation

$$\Psi_{X,Z}(\Gamma' \circ \Gamma) = \Psi_{Y,Z}(\Gamma') \circ \Psi_{X,Y}(\Gamma),$$

where X, Y, Z are smooth projective surfaces, $\Gamma \in A^2(X \times Y)$ and $\Gamma' \in A^2(Y \times Z)$. In the case $X = Y$ the map $\Psi_X = \Psi_{X,X}$ yields an isomorphism of rings

$$(2.2) \quad \Psi_X : A^2(X \times X) / \mathcal{J}(X) \rightarrow \mathrm{End}_{\mathcal{M}_{rat}}(t_2(X))$$

where $\mathcal{J}(X)$ is the ideal of $A^2(X \times X)$ generated by the classes of correspondences which are not dominant over X by either the first or the second projection (see [KMP, 7.4.3]). The following result shows that, if $q(X) = 0$, then $\mathcal{J}(X)$ coincides with the ideal $\mathcal{I}(X)$ of degenerate correspondences.

I(X)=J(X)

Lemma 2.3. *Let X be a smooth projective surface with $q(X) = 0$. Then $\mathcal{I}(X) = \mathcal{J}(X)$ in $A^2(X \times X)$.*

Proof. From the definition of the ideals $\mathcal{J}(X)$ and $\mathcal{I}(X)$ we get $\mathcal{I}(X) \subseteq \mathcal{J}(X)$. Let $\Gamma \in \mathcal{J}(X)$ such that Γ is not dominant over X under the first projection. We claim that Γ belongs to the ideal of degenerate correspondences. Γ vanishes on some $V \times X$, with V open in X , hence it has support on $W \times X$, with $\dim W \leq 1$. If $\dim W = 0$ then $\Gamma = \sum_i n_i [X \times P_i]$ in $A^2(X \times X)$, where P_i are closed points in X . Hence $\Gamma \in \mathcal{I}(X)$. If $\dim W = 1$ then $\Gamma \in A^1(W \times X)$, where $A^1(W \times X) = p_1^*(A^1(W)) \times p_2^*(A^1(X))$, with p_i the projections, because $H^1(X, \mathcal{O}_X) = 0$. Therefore $\Gamma \in \mathcal{I}(X)$. \square

Corollary 2.4. *Let X be a K3 surface. Then the map $\Psi_X : A^2(X \times X) \rightarrow \text{End}_{\mathcal{M}_{\text{rat}}}(t_2(X))$ yields the following isomorphisms*

$$A^2(X \times X)/\mathcal{I}(X) \simeq \text{End}_{\mathcal{M}_{\text{rat}}}(t_2(X)) \simeq A_0(X_{k(X)})/A_0(X)$$

where $A_0(X_{k(X)}) = \varinjlim_{U \subset X} A^2(U \times X)$

Proof. Everything follows from Lemma 2.3 and the isomorphism in [KMP.7.5.10], because $q(X) = 0$. In the isomorphism $\text{End}_{\mathcal{M}_{\text{rat}}}(t_2(X)) \simeq A_0(X_{k(X)})/A_0(X)$ the class $[\xi]$ of the generic point ξ of X in the ring $A_0(X_{k(X)})$ corresponds to the identity map of the motive $t_2(X)$. \square

int-pair

Lemma 2.5. *Let X be a complex K3 surface and let $Z \in A^2(X \times X)$ which acts as 0 on $H^{2,0}(X)$. Then Z acts as 0 on $H_{\text{tr}}^2(X)$.*

Proof. Under the intersection pairing on $H^2(X, \mathbf{C})$ the orthogonal complement of $NS(X) \otimes \mathbf{C}$ is $\pi_2^{\text{tr}} H^2(X)$, where π_2^{tr} is the class of Δ_X in $H_{\text{tr}}^2(X) \times H_{\text{tr}}^2(X) \subset H^4(X \times X)$. This immediately follows from the equality

$$a \cdot b = (a \cdot \pi_2^{\text{tr}}) \cdot (\pi_2^{\text{alg}} \cdot b) = a \cdot (\pi_2^{\text{tr}} \cdot \pi_2^{\text{alg}}) \cdot b = 0$$

where $a \in H_{\text{tr}}^2(X)$ and $b \in NS(X)$, because $\pi_2^{\text{tr}} \cdot \pi_2^{\text{alg}} = 0$. On the other hand

$$NS(X)_{\mathbf{Q}} = H^{1,1} \cap H^2(X, \mathbf{Q}) = \{x \in H^2(X)/x \cdot \omega = 0, \forall \omega \in H^{2,0}(X)\}.$$

Therefore, if Z acts as 0 on $H^{2,0}(X)$ then the orthogonal complement of $\ker(Z)|_{H_{\text{tr}}^2}$ is contained in $NS(X) \otimes \mathbf{C}$, hence Z acts as 0 on $\pi_2^{\text{tr}} H^2(X)$. \square

Definition 2.6. Let X be a smooth projective variety of dimension d over a field k . The *indices* of a correspondence $\Gamma \subset X \times X$ are the numbers $\alpha(\Gamma) = \deg(\Gamma \cdot [P \times X])$ and $\beta(\Gamma) = \deg(\Gamma \cdot [X \times P])$, where P is any rational point on X ; see [Fu, 16.1.4]. The indices are additive in Γ , and $\beta(\Gamma) = \alpha({}^t\Gamma)$.

A correspondence is said of *valence zero* if it belongs to the ideal $\mathcal{I}(X)$ in $A^d(X \times X)$ of degenerate correspondences. A correspondence Γ has *valence* v if $\Gamma + v\Delta_X$ has valence 0. If Γ_1, Γ_2 in $A^d(X \times X)$ have valences v_1, v_2 then $\Gamma = \Gamma_1 + \Gamma_2$ has valence $v_1 + v_2$, and $\Gamma_1 \circ \Gamma_2$ has valence $-v_1 v_2$ by [Fu, 16.1.5(a)]. If $p \in A^d(X \times X)$ is a projector, i.e. $p^2 = p$, which has a valence, then $v(p)$ equals either 0 or -1.

Murre-D

Lemma 2.7. *Let X be smooth projective surface over a field k of characteristic 0 with $q(X) = 0$ and $p_g(X) \neq 0$. Assume that every correspondence $\Gamma \in A^2(X \times X)$ has a valence. Then*

- (i) $A^2(X \times X)_{\text{hom}} \subset \mathcal{I}(X)$;
- (ii) Bloch's conjecture 1.1 holds true.

Proof. Let $\Gamma \in A^2(X \times X)_{hom}$, with $\Gamma + v(\Gamma)\Delta_X \in \mathcal{I}(X)$. Let

$$cl : A^2(X \times X) \rightarrow H^4(X \times X) \simeq \sum_{0 \leq p+q \leq 4} H^p(X) \times H^q(X).$$

Then $cl(\Gamma) = 0$ and $v(\Gamma)cl(\Delta_X) = cl(Z)$, with $Z \in \mathcal{I}(X)$. For every cycle $Z \in \mathcal{I}(X)$ the component of $cl(Z)$ in $H_{tr}^2(X) \otimes H_{tr}^2(X)$ vanishes. Therefore $v(\Gamma)cl(\Delta_X) = 0$ in $H_{tr}^2(X) \otimes H_{tr}^2(X)$. Since $p_g(X) \neq 0$ we have $cl(\Delta_X) \neq 0$, hence $v(\Gamma) = 0$, i.e. $\Gamma \in \mathcal{I}(X)$. It follows that $\Gamma \in \mathcal{I}(X) \cap A^2(X \times X)_{hom}$, which proves (i).

We have $\Gamma = Z_1 + Z_2$, with $Z_1 = \sum_i n_i [P_i \times X]$, $Z_2 = \sum_j m_j [X \times Q_j]$ and $\sum_i n_i = \sum_j m_j = 0$. Let Ω be a universal domain containing k . Then, for every $\alpha \in A_0(X_\Omega)_0$ we have $(Z_1)_*(\alpha) = (Z_2)_*(\alpha) = 0$. Therefore Γ acts as 0 on $A_0(X_\Omega)_0 = T(X_\Omega)$ and this proves that X satisfies Bloch's conjecture 1.1. \square

Remark 2.8. Note that condition (i) above is equivalent to $\text{End}_{\mathcal{M}_{rat}}(t_2(X)) \simeq \text{End}_{\mathcal{M}_{hom}}(t_2(X))$ and implies that every endomorphism of $h(X)$ which is homologically trivial is nilpotent, see [KMP, 7.6.12].

main thm

Theorem 2.9. *Let X be a complex K3 surface. Then X satisfies Bloch's conjecture if and only if every correspondence $\Gamma \in A^2(X \times X)$ has a valence. If $h(X)$ is finite dimensional then all valences belong to \bar{Q} .*

Proof. If every correspondence $\Gamma \in A^2(X \times X)$ has a valence then X satisfies Bloch's conjecture, by 2.7 (ii).

Conversely suppose that X satisfies Bloch's conjecture and let $\Gamma \in A^2(X \times X)$. Let ω be a generator of $H^{2,0}(X)$ as a \mathbf{C} -vector space and let $\Gamma^*(\omega) = \alpha_\Gamma \cdot \omega$, with $\alpha_\Gamma \in \mathbf{C}$. Let $Z = \Gamma - \alpha_\Gamma \Delta_X \in A^2(X \times X) \otimes \mathbf{C}$. Then the correspondence Z acts as 0 on $H^{2,0}(X)$. From 2.5 Z acts as 0 on $\pi_2^{tr} H^2(X)$. Therefore $\bar{Z} = \pi_2^{tr} \circ Z \circ \pi_2^{tr}$ acts as 0 on $H^2(X)$, hence its cohomology class vanishes. From Bloch's conjecture we get that \bar{Z} acts as 0 on $A_0(X_\Omega)_0$, where $\Omega = \mathbf{C}$. Since π_2^{tr} acts as the identity on the group of 0-cycles of degree 0 also Z acts as 0 on $A_0(X_\Omega)$. Let $K = k(X)$ and choose an embedding $\sigma : K \subset \Omega$. σ induces an injective map $A_0(X_K) \rightarrow A_0(X_\Omega)$. Therefore Z acts as 0 on $A_0(X_K)$. Let $[\xi]$ be the class of the generic point ξ of X in the quotient $A_0(X_K)/A_0(X)$. Then $Z_*([\xi]) = 0$. From the isomorphism in Corollary 2.4 we get $Z = \Gamma - \alpha_\Gamma \Delta_X \in \mathcal{I}(X)$ and hence Γ has valence $v(\Gamma) = -\alpha_\Gamma$ in $A^2(X \times X) \otimes \mathbf{C}$.

If $h(X)$ is finite dimensional then $t_2(X)$ is evenly finite dimensional. Let $\Gamma + v(\Gamma)\Delta_X \in \mathcal{I}(X)$, with $v(\Gamma) \in \mathbf{C}$. Then

$$\Psi_X(\Gamma) = -v(\Gamma) \cdot id_{t_2(X)} \in (\text{End}_{\mathcal{M}_{rat}}(t_2(X)) \otimes \mathbf{C},$$

where Ψ_X is the map in (2.2). Let n be such that $\bigwedge^n t_2(X) = 0$. Then there exists a non-zero polynomial $G(T) \in \mathbf{Q}[T]$ of degree $n - 1$ such that the endomorphism $f_\Gamma = -v(\Gamma) \cdot \text{id}_{t_2(X)}$ satisfies $G(f_\Gamma) = 0$, see [MNP, Theorem 5.5.1]. Therefore $-v(\Gamma) \in \bar{\mathbf{Q}}$. \square

In [Vois 2, Cor.3.10] it is proved that, if the motive $h(X)$ of a complex K3 surface X is finite dimensional, then $t_2(X)$ is indecomposable, that is any sub motive of it is either the whole motive or it is the 0-motive. The following Corollary shows that the same result holds true if every correspondence has a valence.

$t_2(X)$

Corollary 2.10. *Let X be complex K3 surface such that every correspondence in $A^2(X \times X)$ has a valence. Then the motive $t_2(X)$ is indecomposable*

Proof. The category \mathcal{M}_{rat} is pseudoabelian, hence a submotive M of $t_2(X)$ is defined by a projector $\pi \in A^2(X \times X)$ such that $\pi \circ \pi_2^{tr} = \pi_2^{tr} \circ \pi = \pi$. π has a valence which may be either 0 or -1. If $v(\pi) = -1$ then π acts as the identity on $H^{2,0}(X)$, hence the correspondence $Z = \pi_2^{tr} - \pi$ acts as 0. By the same argument as in the proof of Theorem 2.9 Z is homologically trivial, hence, by 2.7(i) $Z \in \mathcal{I}(X)$. Therefore π equals the identity of $t_2(X)$. Similarly if $v(\pi) = 0$ then π acts as the 0-map in $\text{End}_{\mathcal{M}_{rat}}(t_2(X))$. Therefore every submodule of $M = t_2(X)$ is either the whole motive or the 0-motive. This proves that $t_2(X)$ is indecomposable. \square

Remark 2.11. Let X be a general complex K3 surface, i.e X is a general member of a smooth projective families $\{X_t\}$ over the disk Δ . For a general complex K3 surface X it is known that the homological motive $t_2^{hom}(X) \in \mathcal{M}_{hom}$ is absolutely simple, i.e it is simple in the category $\mathcal{M}_{hom}(\bar{\mathbf{Q}})$, see [Ped1.5.4]. If every correspondence $\Gamma \in A^2(X \times X)$ has a valence then, by Lemma 2.6 and Remark 2.7,

$$\text{End}_{\mathcal{M}_{hom}}(t_2^{hom}(X)) \simeq \text{End}_{\mathcal{M}_{rat}}(t_2(X)).$$

Therefore $\text{End}_{\mathcal{M}_{rat}}(t_2(X)) \simeq \bar{\mathbf{Q}}$.

3. FINITE GROUP OF AUTOMORPHISMS ON K3 SURFACES

Let X be a complex K3 surface and let g be an automorphisms of X . The global holomorphic 2-forms $H^{2,0}(X)$ has complex dimension 1, i.e. $H^{2,0}(X) \simeq \mathbf{C}\omega$. Since any automorphism $g \in \text{Aut}(X)$ preserves the vector space $H^{2,0}(X)$, there is a non-zero complex number $\alpha(g) \in \mathbf{C}^*$ such that $g_*(\omega) = \alpha(g)\omega$.

Definition 3.1. An automorphism $g \in G$ is *symplectic* if $\alpha(g) = 1$, i.e if g acts as the identity on $H^{2,0}(X)$, while g is non symplectic if $\alpha(g) \neq 1$. If g is non-symplectic and has order m then $\alpha(g) = \zeta$, is a primitive m -th root of unity in \mathbf{C}^*

Nikulin (see [Ni, Thm.3.1]) proved that $\alpha(\text{Aut } X)$ is a finite cyclic group of order m and the Euler function $\phi(m)$ divides the rank of $T_{X,\mathbf{Q}}$. Here $T_{X,\mathbf{Q}} = H_{tr}^2(X, \mathbf{Q})$ denotes the lattice of transcendental cycles on X . $T_{X,\mathbf{Q}}$ can be described as the orthogonal complement of $NS(X)_{\mathbf{Q}}$ in $H^2(X, \mathbf{Q})$. $H_{tr}^2(X, \mathbf{Q})$ is the smallest sub-Hodge structure of $H^2(X, \mathbf{Q})$ containing $H^{2,0}(X)$. Since $H^{2,0}(X) \subset H_{tr}^2(X) \otimes \mathbf{C}$ is compatible with the action of an automorphism g of X , g is symplectic iff it acts as the identity on $H_{tr}^2(X)$.

Let g be of order m and let $p = (1/m) \sum_{0 \leq i \leq m-1} \Gamma_{g^i}$ and $\Psi_X(p) = \pi \in \text{End}_{\mathcal{M}}(t_2(X))$. π is a projector, hence π acts either as 0 or as the identity on $H^{2,0}(X) \simeq \mathbf{C}\omega$. If g is non-symplectic then $g_*(\omega) = \zeta\omega$, with ζ a primitive m -th root of 1, so that π acts as 0 on $H^{2,0}(X)$. If g is symplectic then $g_*(\omega) = \omega$ and π acts as the identity on $H^{2,0}(X)$. In the first case $\pi_* = 0$ on $H_{tr}^2(X, \mathbf{Q})$. In the second case π_* equals the identity on $H_{tr}^2(X, \mathbf{Q})$. The following result has been proved by C.Voisin (see [Vois 1]) in the case of a symplectic involution, and then extended by D.Huybrechts in [Huy 1] to any symplectic automorphism g of finite order.

Theorem 3.2. (*Huybrechts*) *Let g be a symplectic automorphism of finite order on a complex K3 surface X . Then g acts as the identity on $A_0(X)_0$.*

A finite group G of automorphism of the K3 surface X acts on the transcendental motive $t_2(X)$, via the action of the correspondences Γ_g . The following result characterizes the action of symplectic and non-symplectic automorphisms on $t_2(X)$ and proves that the projector $p = (1/n) \sum_{g \in G} \Gamma_g$ has a valence.

Theorem 2

Theorem 3.3. *Let X be a K3 surface over \mathbf{C} with a finite group G of order n of automorphisms. Let Y be a minimal desingularization of the quotient surface X/G . Then*

(i) *If G consists of symplectic automorphisms $t_2(X)^G = t_2(X) = t_2(Y)$ and the rational map $X \rightarrow Y$ induces an isomorphism of motives $h(X) \simeq h(Y)$.*

(ii) *If G consists of non-symplectic automorphisms, then $t_2(X)^G = t_2(Y) = 0$, which implies that $p_g(Y) = 0$.*

The projector

$$p = (1/n) \sum_{g \in G} \Gamma_g$$

has valence $v(p) = -1$, if G consists of symplectic automorphisms, while $v(p) = 0$ if the automorphisms of G are non symplectic.

Proof. (i) From Theorem 3.2 every symplectic automorphism $g \in G$ acts trivially on $A_0(X)$, so that $A_0(X)^G = A_0(X)$. Since G is symplectic there are a finite number $\{P_1, \dots, P_k\}$ of isolated fixed points for G on X . Let Y be a minimal desingularization of the quotient surface X/G . The maps $f : X \rightarrow X/G$ and $\pi : Y \rightarrow X/G$ yield a rational map $X \rightarrow Y$, which is defined outside $\{P_1, \dots, P_k\}$. $t_2(-)$ is a birational invariant for smooth projective surfaces, hence we may blow up X/G to assume that $Y = X/G$. We get a map $\theta : t_2(X) \rightarrow t_2(Y)$, and we claim that θ is the projection onto a direct summand. Let $f : X \rightarrow Y$. Then $\theta = \Psi_{X,Y}(\Gamma_f)$. Γ_f has a right inverse $(1/m)({}^t\Gamma_f)$ and $mp = {}^t\Gamma_f \circ \Gamma_f = \Delta_X + \sum_{g \neq 1} \Gamma_g$. It follows, from the functoriality of the map Ψ , that $t_2(X)^G$ is a direct summand of $t_2(X)$. Let

$$t_2(X) = t_2(X)^G \oplus N.$$

Since $A_i(t_2(X)) = 0$, for $i \neq 0$, and $A_0(X)^G = A_0(X)$ we get $A_i(N) = 0$, for all i . From [GG, Lemma 1] we get $N = 0$, hence $t_2(X) = t_2(X)^G \simeq t_2(Y)$.

The surface Y is a K3 surface and $H_{tr}^2(X, \mathbf{Q}) \simeq H_{tr}^2(Y, \mathbf{Q})$ because every $g \in G$ acts trivially on $H_{tr}^2(X)$. Thus $\rho = \rho(X) = \rho(Y)$. The motives $h(X)$ and $h(Y)$ have Chow-Künneth decomposition as follows

$$h(X) \simeq \mathbf{1} \oplus \mathbf{L}^{\oplus \rho} \oplus t_2(X) \oplus \mathbf{L}^2;$$

$$h(Y) \simeq \mathbf{1} \oplus \mathbf{L}^{\oplus \rho} \oplus t_2(Y) \oplus \mathbf{L}^2.$$

from the isomorphism $t_2(X) \simeq t_2(Y)$ we get $h(X) \simeq h(Y)$.

(ii) Let Y be a desingularization of the quotient surface X/G . We have $t_2(Y) = t_2(X)^G$, because $t_2(X)$ is a birational invariant. Also $q(Y) = 0$ and $p_g(Y) = 0$, see [Ni, Thm.3.1a]. Y is not a surface of general type, hence $A_0(X)_0 = 0$ which implies $t_2(Y) = 0$.

If every $g \in G$ is symplectic then, by (i), Γ_g acts as the identity on $t_2(X)$, for all $g \in G$. From the isomorphism in Cor 2.4 we get $\Gamma_g - \Delta_X \in \mathcal{I}(X)$, for every $g \in G$ i.e $v(\Gamma_g) = -1$. Therefore $v(p) = -1$. If G consists of non-symplectic automorphisms, then, by (ii), $t_2(Y) = t_2(X)^G = 0$, and hence p acts as 0 on $t_2(X)$. Therefore $p \in \mathcal{I}(X)$ and $v(p) = 0$. \square

Next we apply Theorem 3.3 to compute the virtual number of coincidences for the projector p associated to a finite group of automorphisms

on a K3 surface, using a formula by Severi , as reconstructed in [PW 1,p 499].

Theorem 3

Theorem 3.4 (Severi's formula). *Let X be a smooth projective surface. If Δ_X does not have valence 0 and $\Gamma \in A^2(X \times X)$ is a correspondence with valence v , then*

$$\deg(\Gamma \cdot \Delta_X) = \alpha(\Gamma) + \beta(\Gamma) + t(\Gamma) + v \cdot (2 + \rho(X) - c_2(X)).$$

where $t(\Gamma)$ is the trace of the action of Γ on the finite dimensional vector space $NS(X) \otimes \mathbf{Q}$.

p and q

Corollary 3.5. *Let X be a K3 surface with a finite group G of automorphism of order n . Let $p = (1/n) \sum_{g \in G} \Gamma_g$. If G consists of symplectic automorphisms then*

$$(3.6) \quad \deg(p \cdot \Delta_X) = 2 + t(p) + 22 - \rho(X) = 24 + t(p) - \rho(X)$$

while, if the automorphisms of G are non-symplectic,

$$(3.7) \quad \deg(p \cdot \Delta_X) = 2 + t(p)$$

where $t(p)$ is the trace of the action of p on the finite dimensional vector space $NS(X) \otimes \mathbf{Q}$.

Proof. From Theorem 3.3 the projector p has valence $v(p) = -1$ in the case G consists of symplectic automorphisms while $v(p) = 0$ in the non-symplectic case. Therefore everything follows from Theorem 3.4, because $c_2(X) = 24$ and the correspondence p has indices 1. \square

Example 3.8. The following example shows how the equality in (3.7) can be used to recover some results, obtained by Nikulin in [Ni], for symplectic automorphisms of prime order on a K3 surface.

Let X be a K3 surface with a symplectic automorphism g of order a prime number l . Let G be the group of order l generated by g and let $t(g)$ be the trace of the action of g on the finite dimensional vector space $NS(X)_{\mathbf{Q}}$. From Theorem 3.3 g acts as the identity on $t_2(X)$ hence the correspondence $\Gamma_g \in A^2(X \times X)$ has valence -1. Let k be the number of isolated fixed points of g . All the fixed points $x \in X$, under the action of G , have the same order l , because the stationary subgroup G_x of x , being a cyclic subgroup of G , coincides with G . Therefore from (3.7) we get

$$(3.9) \quad k = \deg(\Gamma_g \cdot \Delta_X) = 24 + t(g) - \rho(X)$$

i.e. $t(g) = k + \rho(X) - 24$, for every $g \in G$, $g \neq 1$, where $t(g) = \text{tr}_{NS(X)}(g)$.

Let $p = (1/l) \sum_{g \in G} \Gamma_g$ the projector in $A^2(X \times X)$. Since all automorphisms $g^i \in G$, with $i \neq 0$, have the same number of fixed points we get

$$t(p) = tr_{NS(X)}(p) = 1/l(\rho(X) + (l-1)t(g)).$$

Let $M = (X, p) \in \mathcal{M}_{rat}$: then $h(X) = M \oplus N$, with $N = (X, 1-p)$ and

$$A^1(h(X)) = NS(X)_{\mathbf{Q}} = A^1(M) \oplus A^1(N).$$

where $A^1(M) = p^* A^1(X) \simeq NS(X)_{\mathbf{Q}}^G$ and $A^1(N) = (1-p)^* A^1(X)$. The projector p acts as 0 on $A^1(N)$, hence

$$t(p) = tr_{NS(X)}(p) = tr_{NS(X)_{\mathbf{Q}}^G}(p) = s = 1/l(\rho(X) + (l-1)t(g))$$

where $s = \dim_{\mathbf{Q}}(NS(X)_{\mathbf{Q}}^G)$. Therefore

$$(3.10) \quad t(g) = \frac{ls - \rho(X)}{l-1}$$

Here $s \geq 1$ because every automorphism fixes a hyperplane section of X .

Let Y be a minimal resolution of the singular points of X/G . Then Y is a K3 surface and, by Theorem 3.3, $h(X) \simeq h(Y)$. In particular

$$A^1(h(X)) \simeq A^1(h(Y)) \simeq \mathbf{L}^{\oplus \rho(X)}.$$

Let π be the rational map obtained by composing the morphisms $X \rightarrow X/G$ and $Y \rightarrow X$. π is defined outside of a finite number of points in X . Therefore π induces a map $NS(X)_{\mathbf{Q}} \rightarrow NS(Y)_{\mathbf{Q}}$ where

$$NS(X)_{\mathbf{Q}} = NS(X)_{\mathbf{Q}}^G \oplus A^1(N) ; \quad NS(Y)_{\mathbf{Q}} = NS(X)_{\mathbf{Q}}^G \oplus M_Y.$$

Here M_Y is the subvector space of $NS(Y)_{\mathbf{Q}}$ generated by the classes of the divisors corresponding to the curves obtained by resolving the singularities of X/G . All the fixed points of G on X have order l and, for every fixed point x , the minimal resolution gives a curve consisting of $(l-1)$ rational curves. Therefore M_Y has dimension $k(l-1)$ and we get

$$(3.11) \quad \rho(X) = \rho(Y) = \dim_{\mathbf{Q}}(NS(X)_{\mathbf{Q}}^G) + k(l-1) = s + k(l-1)$$

The equalities in (3.9), (3.10) and (3.11) give

$$k = 24/l + 1$$

We also have $l \leq 7$ because from (3.11) $l = 11$ implies $\rho(X) \geq 21$ which is impossible. Note that, by a result of Nikulin in [NI], the order of a symplectic automorphism is always ≤ 8 . In [GS,Th.0.1] it is proved that a K3 surface cannot have a symplectic and a non-symplectic automorphism of the same order $l = 7$.

4. CONSTANT CYCLE CURVES ON K3 SURFACES

Let X and Y be a smooth projective varieties over a field k and let $h(X)$ and $h(Y)$ be their motives in $\mathcal{M}_{rat}(k)$. If we pass to the (covariant) category $\mathcal{M}_{rat}^o(k)$ of birational motives (see [KMP, 14,7,5]) the Lefschetz motive \mathbf{L} goes to zero. Writing $\bar{h}(X)$ for the birational motive of a smooth projective variety X of dimension d , and Hom_{bir} for morphisms in $\mathcal{M}_{rat}^o(k)$, the fundamental formula is:

$$A_0(X_{k(Y)}) = \varinjlim_{U \subset Y} A^d(U \times X) \cong \text{Hom}_{bir}(\bar{h}(Y), \bar{h}(X)).$$

In particular, $A_0(X) \cong \text{Hom}_{bir}(\mathbf{1}, \bar{h}(X))$ for all X . Let X be a smooth projective surface, C a smooth projective curve. Let $h(C) = \mathbf{1} \oplus h_1(C) \oplus \mathbf{L}$ and $h(X) = \mathbf{1} \oplus h_1(X) \oplus h_2^{alg}(X) \oplus t_2(X) \oplus h_3(X) \oplus \mathbf{L}^2$ be Chow-Künneth decomposition for the motives of X and C . Since the birational motive of a curve is $\mathbf{1} \oplus \bar{h}_1(C)$ and the birational motive of a surface X is $\mathbf{1} \oplus \bar{h}_1(X) \oplus \bar{t}_2(X)$ we get

$$A_0(X_{k(C)}) \simeq A_0(X) \oplus \text{Hom}_{bir}(\bar{h}_1(C), \bar{h}_1(X)) \oplus \text{Hom}_{bir}(\bar{h}_1(C), \bar{t}_2(X))$$

If $q(X) = 0$ then $h_1(X) = h_3(X) = 0$. Moreover $h_1(C)$ and $t_2(X)$ are birational motives, see [KMP, 148.8]. Therefore

$$(4.1) \quad A_0(X_{k(C)}) \cong A_0(X) \oplus \text{Hom}_{\mathcal{M}_{rat}}(h_1(C), t_2(X))$$

Remark 4.2. The isomorphism in (4.1) depends only on $k(C)$, hence it does not change if we replace C with any other curve which is a smooth projective model of $k(C)$.

Let X be a smooth projective surface with $q(X) = 0$ and let C be a smooth curve. Consider the map

$$A_1(C \times X) \rightarrow \varinjlim_{U \subset C} A_1(U \times X) \simeq A_0(X_{k(C)}),$$

where $k(C) = k(\eta_C)$, with η_C the generic point of C and let

$$\Psi_{C,X} : A_1(C \times X) \rightarrow \frac{A_0(X_{k(C)})}{A_0(X)} \simeq \text{Hom}_{\mathcal{M}_{rat}}(h_1(C), t_2(X))$$

be the composite map .

lem:Psi

Lemma 4.3. *Let $\mathcal{I}(C, X)$ denote the subgroup of $A_1(C \times X)$ generated by the classes of degenerate correspondences, i.e. correspondences of the form $P \times D$ and $C \times Q$, where $P \in C$ and $Q \in X$ are closed points and D is a curve on X .*

The map $\Psi_{C,X}$ induces an isomorphism

$$\frac{A_1(C \times X)}{\mathcal{I}(C, X)} \simeq \text{Hom}_{\mathcal{M}_{rat}}(h_1(C), t_2(X)).$$

Proof. The homology class of every $Z \in A_1(C \times X)$ lies in $H_2(C) \oplus H_2(X)_{alg}$, so that adding to Z vertical and horizontal cycles, which belong to $\mathcal{I}(C, X)$, we get a cycle Z' homologous to zero such that $Z = Z' \in A_1(C \times X)/\mathcal{I}(C, X)$. So we may assume that $Z \in A_1(C \times X)_{hom}$. The correspondences $P \times D$ generate the kernel of $A_1(C \times X) \rightarrow A_0(X_{k(C)})$. The correspondence $C \times Q$ maps to the image of Q under $A_0(X) \rightarrow A_0(X_{k(C)})$. Therefore $\mathcal{I}(C, X)$ is in the kernel of the composition

$$\Psi_{C,X} : A_1(C \times X) \rightarrow A_0(X_{k(C)}) \rightarrow \text{Hom}_{\mathcal{M}_{rat}}(h_1(C), t_2(X))$$

Conversely, let $Z \in A_1(C \times X)_{hom}$ such that its class $\Psi_{C,X}(Z)$ in $A_0(X_{k(C)})$ is in the subgroup $A_0(X)$. Then there are closed points Q_i of X such that $\Psi_{C,X}(Z) = \sum n_i \Psi_{C,X}(C \times Q_i)$. Subtracting $\sum n_i [C \times Q_i]$ from Z , we may assume that $\Psi_{C,X}(Z) = 0$. But then Z vanishes in $A_1(U \times X)$ for some $U = C - \{P_j\}$ and hence Z is a linear combination of cycles of the form $P_j \times D_j$. \square

The following definition has been given in [Huy 2]

Definition 4.4. Let X be a smooth projective surface over an algebraically closed field k , C an integral curve on X and let $f : C \rightarrow X$ be a closed immersion. C is called a *constant cycle curve* if the class $[\eta_C]$ belongs to the image of $f_* : A_0(X) \rightarrow A_0(X_{k(C)})$. Here η_C is the generic point of C and $[\eta_C]$ is viewed as a closed point of the surface $X_{k(C)}$ over the field $k(C)$.

Remark 4.5. Every rational curve C is a constant cycle curve. In the above definition, by eventually taking $k(\tilde{C}) = k(C)$, where \tilde{C} is a desingularization of C , we can assume C to be smooth. If X is defined over \mathbf{C} , then by [Huy 2, 3.7] a curve $f : C \rightarrow X$ is a constant cycle curve iff the induced map $f_* : A_0(C)_0 \rightarrow A_0(X)_0$ vanishes.

If X is a K3 surface over \mathbf{C} then a curve C is a constant cycle curve iff any point on C is rationally equivalent to c_X , see [Vois 3, 2.2].

For every K3 surface X over an algebraically closed field k , c_X is the distinguished class of degree one in $A_0(X)$, introduced in [BV]. c_X is the class of any closed point $P \in X$ lying on rational curve R . Because for any irreducible curve C on X there is a rational curve $R \neq C$ which intersect C , we can represent c_X by the class of a point $P \in C$, namely any point of $C \cap R$.

The following result gives a condition on the motives of C and X in order for C to be constant cycle curve, in the case of a surface X with $q(X) = 0$.

Theorem 4

Theorem 4.6. *Let X be a smooth projective surface over a field k , with $q(X) = 0$, and let $f : C \rightarrow X$ be a curve on X . The following conditions are equivalent*

- (i) *C is a constant cycle curve;*
- (ii) *The map $f : C \rightarrow X$ induces the 0- map f_* in $\text{Hom}_{\mathcal{M}_{rat}}(h_1(C), t_2(X))$.*

Proof. From the commutative diagram

$$\begin{array}{ccc} A_1(C \times C) & \xrightarrow{f_*} & A_1(C \times X) \\ \downarrow & & \downarrow \Psi_{C,X} \\ \frac{A_0(C_{\eta_C})}{A_0(C)} & \xrightarrow{\bar{f}_*} & \frac{A_0(X_{k(C)})}{A_0(X)} \end{array}$$

where $\bar{f}_*([\eta_C]) = [\eta_C] \in A_0(X_{k(C)})$, we get $\Psi_{C,X}(f_*(\Delta_C)) = [\eta_C]$. Therefore $[\eta_C] = 0$ in $\frac{A_0(X_{k(C)})}{A_0(X)}$ iff $f_*(\Delta_C) \in \mathcal{I}(C, X)$. From Lemma 4.3, it follows that $f_*(\Delta_C) \in \mathcal{I}(C, X)$ iff the map induced by f in $\text{Hom}_{\mathcal{M}_{rat}}(h_1(C), t_2(X))$ vanishes. Therefore (i) \Leftrightarrow (ii). \square

Corollary 4.7. *Let X be a complex K3 surface with a symplectic automorphism g of finite order n . Let Y be the minimal desingularization of X/g . Then there is a 1-1 correspondence between constant cycle curves on X fixed by g and constant cycle curves on Y .*

Proof. Let $f : C \rightarrow X$ be a constant cycle curve on X . The maps $\pi : X \rightarrow X/g$ and $Y \rightarrow X/g$ yield a rational map $X \rightarrow Y$, which is defined outside a finite number of points on X . $t_2(X)$ and $h_1(C)$ are birational invariants, hence we may blow up X/g to assume that $X/g = Y$ and $\pi : X \rightarrow Y$. Then Y is a K3 surface and the finite map π induces an isomorphism between $t_2(X)$ and $t_2(Y)$, as in the proof of Theorem 3.3. From Theorem 4.6 the map $f_* \in \text{Hom}_{\mathcal{M}_{rat}}(h_1(C), t_2(X))$ vanishes, hence also $\tilde{f}_* = 0$, with $\tilde{f} = \pi \circ f$. Therefore $\tilde{f} : C \rightarrow Y$ is a constant cycle curve.

Conversely let $D \subset Y$ be a constant cycle curve which is the image of a curve C on X fixed by g . The map π induces group homomorphisms

$$\pi_* : A_0(X_{k(C)})/A_0(X) \rightarrow A_0(Y_{k(D)})/A_0(Y);$$

$$\pi^* : A_0(Y_{k(D)})/A_0(Y) \rightarrow A_0(X_{k(C)})/A_0(X).$$

Let G be the cyclic group generated by g . Then $(\pi^* \circ \pi_*)(\alpha) = \sum_{g \in G} g_*(\alpha)$ for every $\alpha \in A_0(X_{k(C)})/A_0(X)$. From Corollary 2.4 and Theorem 3.3 we get the following isomorphisms

$$A_0(X_{k(X)})/A_0(X) \simeq \text{End}_{\mathcal{M}_{rat}}(t_2(X)) \simeq \text{End}_{\mathcal{M}_{rat}}(t_2(Y)) \simeq A_0(Y_{k(Y)})/A_0(Y)$$

where the class of the generic point ξ of X , which corresponds to the identity map of $t_2(X)$, is mapped to the class $[\zeta]$ of the generic point ζ of Y . Therefore we get a commutative diagram

$$\begin{array}{ccc} \frac{A_0(X_{k(X)})}{A_0(X)} & \xrightarrow{\simeq} & \frac{A_0(Y_{k(Y)})}{A_0(Y)} \\ \downarrow & & \downarrow \\ \frac{A_0(X_{k(C)})}{A_0(X)} & \xrightarrow{\pi_*} & \frac{A_0(Y_{k(D)})}{A_0(D)} \end{array}$$

where the vertical maps are induced by the specialization maps $A_0(X_{k(X)}) \rightarrow A_0(X_{k(C)})$ and $A_0(Y_{k(Y)}) \rightarrow A_0(Y_{k(D)})$, which send $[\xi]$ to $[\eta_C]$ and $[\zeta]$ to $[\eta_D]$. Therefore $\pi_*([\eta_C]) = [\eta_D]$. As g acts as the identity on $t_2(X)$, we get $g_*([\xi]) = [\xi]$. The specialization map sends $[\xi]$ to $[\eta_C]$. Therefore $g_*([\eta_C]) = [\eta_C]$ in $A_0(X_{k(C)})/A_0(X)$, and this implies

$$(\pi^* \circ \pi_*)([\eta_C]) = \sum_{g \in G} g_*([\eta_C]) = n[\eta_C] \in A_0(X_{k(C)})/A_0(X).$$

D being a constant cycle curve on Y , $[\eta_D]$ is in the image of the map $A_0(Y) \rightarrow A_0(Y_{k(D)})$, and hence $\pi_*([\eta_C]) = [\eta_D] = 0$ in $A_0(Y_{k(D)})/A_0(D)$. It follows that

$$n[\eta_C] = (\pi^* \circ \pi_*)([\eta_C]) = \pi^*([\eta_D]) = 0$$

in $A_0(X_{k(C)})/A_0(X)$. Therefore C is a constant cycle curve. \square

Example 4.8. Let X be a complex K3 surface, with $\rho(X) = 9$, which is the intersection of 3 quadrics in \mathbf{P}^5 , having a symplectic involution σ , as described in [VG-S, 3.5]. The desingularization Y of X/σ is a quartic surface in \mathbf{P}^3 , and $t_2(X) = t_2(Y)$, by Theorem 3.3. Let $\beta : \tilde{X} \rightarrow X$ be the blow up at the 8 fixed points of σ and let $C = (1/2) \sum_{1 \leq i \leq 8} C_i$ be the corresponding divisor on Y , with C_i a rational curve. Let $f : \tilde{X} \rightarrow Y$ and let $L \in NS(X)$, with $L^2 = 8$, be the line bundle that gives the embedding $\Phi_L : X \rightarrow \mathbf{P}^5$. There is a line bundle $M \in NS(Y)$ such that $\beta^*L = f^*M$, with $M^2 = 4$ and $h^0(M - C) = 2$. Note that $NS(Y) \simeq NS(X)^\sigma \oplus \{[C_1], \dots, [C_8]\}$, where $NS(X)^\sigma$ has rank 1, because $\rho(X) = \rho(Y) = 9$. Let $\mu = \Phi_{M-C} : Y \rightarrow \mathbf{P}^1$ be the map associated to the pencil $|M - C|$. $Y \rightarrow \mathbf{P}^1$ is an elliptic fibration, because $(M - C)^2 = 0$ implies that all curves in the linear system have genus 1. Let $C_0 \subset Y$ be the 0-section of μ and let C_n be the closure of the set of n -torsion points on the smooth fibres Y_t , $t \in \mathbf{P}^1$. Then, by the results in [Huy 2, 6.2], C_n is a constant cycle curve. From Corollary 4.7 every C_n pullbacks to a constant cycle curve on X .

In [Huy 2, 7.1] it is proved that, for a complex K3 surface X , every fixed curve of a non-symplectic automorphism g of finite order is a constant

cycle curve. The following theorem extends Huybrecht's result to every correspondence $\Gamma \in A^2(X \times X)$ on a K3 surface X , such that Γ has a valence $\neq -1$.

Theorem 4.9. *Let X be a complex K3 surface and let $\Gamma \in A^2(X \times X)$. Suppose that Γ has a valence $v(\Gamma)$, with $v(\Gamma) \neq -1$. If $f : C \rightarrow X$ is a curve on X such that $\Gamma_*([x]) = [x]$ in $A_0(X)$, for every $x \in C$, then C is a constant cycle curve.*

Proof. From the isomorphism $\text{Hom}_{\mathcal{M}_{\text{rat}}}(\mathbf{1}, t_2(X)) \simeq A_0(X)_0$ it follows that every 0-cycle α of degree 0 corresponds to a map $t_\alpha : \mathbf{1} \rightarrow t_2(X)$. The action of the correspondence Γ on $\alpha \in A_0(X)_0$ coincides with the composition map $\bar{\Gamma} \circ t_\alpha : \mathbf{1} \rightarrow t_2(X) \rightarrow t_2(X)$, where

$$\bar{\Gamma} = \Psi_X(\Gamma) \in \text{End}_{\mathcal{M}_{\text{rat}}}(t_2(X)) \simeq \frac{A_0(X_{k(X)})}{A_0(X)}.$$

The map $\bar{\Gamma}$ corresponds, in the isomorphism above, to the cycle class $\Gamma([\xi]) = -v(\Gamma)[\xi]$, because $\Gamma + v(\Gamma)\Delta_X \in \mathcal{I}(X)$. Here ξ is the generic point of X and the class $[\xi]$ corresponds to the identity map of $t_2(X)$. Therefore $\bar{\Gamma} \circ t_\alpha$ is multiplication by $-v(\Gamma)$ and $\Gamma_*(\alpha) = -v(\Gamma)\alpha$ in $A_0(X)_0$. Now let x be any point on C and let $y \in C$ be such that $[y] = c_X$ in $A_0(X)$. Then $\Gamma_*([x]) = [x]$ and $\Gamma_*([y]) = [y] = c_X$. Therefore $\Gamma_*(\alpha) = \alpha$, with $\alpha = [x] - c_X \in A_0(X)_0$. We get $(v(\Gamma) + 1)\alpha = 0$ and hence $\alpha = 0$, because $v(\Gamma) \neq -1$. This proves that, for every point $x \in C$ the 0-cycle $[x] - c_X$ vanishes in $A_0(X)_0$, i.e. C is a constant cycle curve. \square

Example 4.10. (1) The result In Theorem 4.8 applies to the case of the fixed locus of the *bitangent correspondence*, considered in [Huy 2]. Let $X \subset \mathbf{P}^3$ be a smooth quartic not containing a line. Let x a generic point on X and let $C_x = T_x X \cap X$. There are six lines passing through x and such that they are bitangent to C_x at some other points $y_{x,1}, \dots, y_{x,6} \in C_x$. The points $\{y_{x,1}, \dots, y_{x,6}\}$ are rationally equivalent in $A_0(X)$. Let $\Gamma \in A^2(X \times X)$ be the bitangent correspondence $x \rightarrow \{y_{x,1}, \dots, y_{x,6}\}$. Then $\Gamma \circ \Gamma = \Delta_X$, because if l_x is a bitangent through x and y_x then $l_x = l_y$ is also the bitangent through $y = y_x$ and $x_y = x$. Therefore Γ_* acts as involution on $A_0(X)$. By [Huy 2,8.1] Γ_* acts as -1 on $A_0(X)_0$, and hence $\Gamma_*([\xi]) = -[\xi]$ in $A_0(X_{k(X)})/A_0(X) \simeq \text{End}_{\mathcal{M}_{\text{rat}}}(t_2(X))$. Therefore $\Gamma + \Delta_X \in \mathcal{I}(X)$, i.e. the correspondence Γ has a valence $v(\Gamma) = 1$. From theorem 4.8 the curve C of contact points of hyperflexes, which is the fixed locus of Γ , is a constant cycle curve.

(2) Let σ be an involution on a K3 surface X such that $t_2(Y) = 0$, where Y is the desingularization of X/σ . Then, by [Ped 2, Theorem 1],

$v(\Gamma_\sigma) = 1$, and hence every curve in the fixed locus of σ is a constant cycle curve. In the case X is a double cover of \mathbf{P}^2 branched over a sextic curve $C \subset X$, then C is a constant cycle curve, because $t_2(\mathbf{P}^2) = 0$. If X is a double cover of a quadric $Q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ in \mathbf{P}^3 , as in [VG-S, 3.5], then double cover is ramified on a curve B of bidegree(4, 4) and the branch curve of the covering involution σ on X has genus 9. C is a constant cycle curve, because $t_2(Q) = 0$ implies that $v(\Gamma_\sigma) = 1$.

5. FAMILIES OF K3 SURFACES AND THE GENERALIZED FRANCHETTA'S CONJECTURE

Franchetta's conjecture on line bundles over the universal curve on the moduli space of curves of genus g , which has been proved in [AC], may be stated as follows.

Conjecture 5.1. (*Franchetta's conjecture*) *Let \mathcal{C} be the universal curve over the function field of the moduli space \mathcal{M}_g of curves of genus g . Then any line bundle on \mathcal{C} is the integral multiple of the canonical bundle, i.e.*

$$(5.2) \quad \text{Pic}(\mathcal{C}/\mathcal{M}_g) \simeq \mathbf{Z}\omega_{\mathcal{C}}/\mathcal{M}_g$$

O.Grady in [O'Gr, 5.3] has asked the following question, which is similar to Franchetta's conjecture, for the universal family of K3 surfaces.

Question 5.3. (*Generalized Franchetta's conjecture*) *Let $g \geq 3$ and let \mathcal{K}_g be the moduli space of complex K3 surfaces with a polarization of degree $(2g - 2)$. By restricting to the open dense subset $S_g = \mathcal{K}_g^0$ parametrising polarized K3 surfaces with trivial automorphism group, we may assume that the family $f : \mathcal{X}_g \rightarrow S_g$ is smooth. Let $\mathcal{Z} \in \text{CH}^2(\mathcal{X})$. Is it true that $\mathcal{Z}|_{X_s} \in \mathbf{Z}c_{X_s}$ for all closed points $s \in S_g$?*

Let $f : \mathcal{X} \rightarrow S$ be a projective family of surfaces over an algebraically closed field k of characteristic 0, i.e. f is projective, and the fibres are surfaces. Let S be smooth of dimension n and let $U = S - Y$, with Y closed in S . Let $i : Y \rightarrow S$ and $j : U \rightarrow S$ be the inclusions. Let $\bar{\mathcal{X}} = \mathcal{X} \times_S Y$, $\mathcal{X}_U = \mathcal{X} \times_S U$. Then there is an exact sequence

$$(5.4) \quad \text{CH}_n(\bar{\mathcal{X}}) \xrightarrow{i_*} \text{CH}^2(\mathcal{X}) \xrightarrow{j^*} \text{CH}^2(\mathcal{X}_U) \longrightarrow 0$$

where i_* and j^* denote push-forward and pull-back induced by the inclusions i and j , see [Fu, 20.3]. There also is a Gysin homomorphism

$$i^! : \text{CH}^2(\mathcal{X}) \rightarrow \text{CH}^2(\bar{\mathcal{X}})$$

and a specialization map $\sigma : \mathrm{CH}^2(\mathcal{X}_U) \rightarrow \mathrm{CH}^2(\bar{\mathcal{X}})$, such that $\sigma(j^*(\alpha)) = i^!(\alpha)$ for all $\alpha \in A^2(\mathcal{X})$. Let X_η be the generic fibre of f . Then

$$(5.5) \quad \lim_{U \subset S} \mathrm{CH}^2(\mathcal{X}_U) = \lim_{U \subset S} \mathrm{CH}^2(\mathcal{X} \times_S U) \simeq \mathrm{CH}^2(\mathcal{X}_{k(S)}) = \mathrm{CH}^2(X_\eta)$$

where $k(S)$ is the quotient field of S and η the generic point of S .

The following result gives a condition equivalent to the generalized Franchetta's conjecture.

Theorem 5.6. *Let $f : \mathcal{X} \rightarrow S$ be a smooth projective family of K3 surfaces over an algebraically closed field k of characteristic 0, with S smooth of dimension n . Let η be the generic point of S , X_η the generic fiber of f and let $X_s = f^{-1}(s)$, for every closed point $s \in S$. Then the following conditions are equivalent.*

(i) $\mathrm{CH}_0(X_\eta)_{\mathbf{Q}} = A_0(X_\eta) \simeq \mathbf{Q}$.

(ii) *There is a distinguished cycle $\mathcal{C} \in A^2(\mathcal{X}) = \mathrm{CH}^2(\mathcal{X})_{\mathbf{Q}}$ such that, for every $\mathcal{Z} \in A^2(\mathcal{X})$, $\mathcal{Z} \in \mathbf{Q}[\mathcal{C}]$, modulo vertical cycles, and*

$$\mathcal{Z}|_{X_s} \in \mathbf{Q}[c_{X_s}]$$

for every closed fiber X_s .

Proof. (i) \implies (ii). By performing a base change we may assume that there exists a section $\pi : S \rightarrow \mathcal{X}$ of f such that $\pi(s)$ represents c_{X_s} in $A^2(X_s)$, for every $s \in S$. Let \mathcal{C} be the class in $A^2(\mathcal{X})$ of $(p_2)_*(\Gamma_\pi)$, where $\Gamma_\pi \subset A_{\dim S}(S \times \mathcal{X})$ and $p_2 : S \times \mathcal{X} \rightarrow \mathcal{X}$. For every closed point $s \in S$ the exact sequence in (5.4), with $Y = \{s\}$ and $U = S - Y$ yields

$$A_n(X_s) \xrightarrow{i_*} A^2(\mathcal{X}) \xrightarrow{j^*} A^2(\mathcal{X}_U) \longrightarrow 0$$

The specialization map $\sigma : A^2(X_U) \rightarrow A^2(\bar{\mathcal{X}})$ induces $\sigma : A^2(X_\eta) \rightarrow A^2(X_s)$. Then $i^!(\mathcal{C}) = c_{X_s} \in A^2(X_s)$. Let α be a generator of $A_0(X_\eta)$ and let $a \cdot \alpha$, with $a \in \mathbf{Q}$, be the class of $j^*(\mathcal{C})$ in $A_0(X_\eta)$ under the map in (5.4). Then $\sigma(a \cdot \alpha) = i^!(\mathcal{C}) = c_{X_s}$, i.e. $\sigma(\alpha) = 1/ac_{X_s}$. Let $\mathcal{Z} \in A^2(\mathcal{X})$; then $i^!(\mathcal{Z}) = \mathcal{Z}|_{X_s} \in A_0(X_s)$ and $\mathcal{Z}|_{X_s} = \sigma(Z_\eta)$, where $Z_\eta = b \cdot \alpha$ is the class of $j^*(\mathcal{Z})$ in $A_0(X_\eta)$. Therefore $\sigma(Z_\eta) = b \cdot \sigma(\alpha) = (b/a)c_{X_s}$ that proves

$$\mathcal{Z}|_{X_s} \in \mathbf{Q}[c_{X_s}].$$

We also have $j^*(\mathcal{Z} - (b/a)\mathcal{C}) = 0$ in $A^2(X_\eta)$, hence $(\mathcal{Z} - (b/a)\mathcal{C}) = i_*(\beta_s)$ with $\beta_s \in A_n(X_s)$. Here $A_n(X_s) = 0$ for $n > 2$.

(ii) \implies (i). Let $\alpha \in A_0(X_\eta)_0$ and let $\mathcal{Z} \in A^2(\mathcal{X})$ be such that $j^*(\mathcal{Z}) = \alpha \in A_0(X_\eta)$. Let $\mathcal{Z} = m\mathcal{C} + \beta$, with $m \in \mathbf{Q}$ and $j^*(\beta) = 0$. Then $i^!(\mathcal{Z}) = mc_{X_s}$ in $A_0(X_s)$, for every closed fibre X_s . $\sigma(\alpha)$ is a 0-cycle of degree 0 in $A_0(X_s)$, under the specialization map $\sigma : A_0(X_\eta) \rightarrow A_0(X_s)$,

because $\deg(\alpha) = 0$. Therefore we get $m = 0$, $\mathcal{Z} = \beta$ in $A^2(\mathcal{X})$ and $j^*(\mathcal{Z}) = \alpha = 0$. \square

The following Corollary of Theorem 5.6 shows tha condition (i) in Theorem 5.6 holds true, for $g = 3, 4, 5$, i.e. In the cases where the projective model in \mathbf{P}^g of a general K3 surface of genus g is a complete intersection of $g - 2$ hypersurfaces.

Corollary 5.7. *Let $f : \mathcal{X} \rightarrow S$ be the family of polarized K3 surfaces of genus g , with $g = 3, 4, 5$. Then $A_0(X_\eta) \simeq \mathbf{Q}$, with X_η the generic fibre of f .*

Proof. Let $g = 3$. The universal family of K3 surfaces of genus $g = 3$ coincides with the family of quartic surfaces in \mathbf{P}^3 . Let $f : \mathcal{X} \rightarrow |\mathcal{O}(4)| = S$ be the family of all quartic surfaces in $\mathbf{P}_{\mathbf{C}}^3$. Then \mathcal{X} is a projective \mathbf{P}^r -bundle over \mathbf{P}^3 , with $r + 1 = 35$. As such \mathcal{X} has a finite dimensional motive, because its motive is isomorphic to the direct sum of copies of the motive of \mathbf{P}^3 . Also the Chow ring of \mathcal{X} is a finitely generated module over $A(\mathbf{P}^3)$ and hence it is a finite dimensional \mathbf{Q} -vector space. Therefore $A^2(\mathcal{X}_U)$, where U is a Zariski open subset of S , is a finite dimensional \mathbf{Q} -vector space and hence

$$A_0(X_\eta) = \lim_{U \subset S} A^2(\mathcal{X}_U) \simeq \mathbf{Q}$$

Let $g = 4$. Then the generic K3 of genus g is the intersection of a smooth quadric Q and a cubic in \mathbf{P}^4 . Let $S = |\mathcal{O}_Q(3)|$ and let $f : \mathcal{X} \rightarrow S$. Then \mathcal{X} is a projective bundle over Q , with $\mathbf{Q} \simeq \mathbf{P}^1 \times \mathbf{P}^1$. Therefore, as in the previous case, the Chow ring of \mathcal{X} is a finitely generated module over $A(\mathbf{P}^1 \times \mathbf{P}^1)$ and hence it is a finite dimensional \mathbf{Q} -vector space. It follows that $A^2(\mathcal{X}_U)$, for U open in S is a finite dimensional \mathbf{Q} vector space and $A_0(X_\eta) \simeq \mathbf{Q}$.

$g = 5$. The generic K3 is the intersection of 3 quadrics in \mathbf{P}^5 and hence it is the base locus of a 3-dimensional linear subspace of $|\mathcal{O}_{\mathbf{P}^5}(2)|$. Let \mathcal{X} be the incidence variety $Z \subset \text{Gr}(3, H^0(\mathcal{O}_{\mathbf{P}^5}(2))) \times \mathbf{P}^5$ given by couples (P, x) , where $x \in \mathbf{P}^5$ belongs to the intersection of the quadrics parametrized by $P \in \text{Gr}(3, H^0(\mathcal{O}_{\mathbf{P}^5}(2)))$. Let $S = \text{Gr}(3, H^0(\mathcal{O}_{\mathbf{P}^5}(2)))$ and let $f : Z \rightarrow S$ be the map induced by the first projection. For every $s \in S$ the fibre $f^{-1}(s) \subset Z$ is the K3 surface of all points in \mathbf{P}^5 lying on the 3 quadrics parametrized by s . All Chow groups $A^k(\text{Gr}(3, H^0(\mathcal{O}_{\mathbf{P}^5}(2))))$ of the Grassmanian bundle $\text{Gr}(3, H^0(\mathcal{O}_{\mathbf{P}^5}(2)))$ are isomorphic to a finite direct sum of Chow groups of \mathbf{P}^5 (see [Fu 14.6.5]) and hence they are finite dimensional \mathbf{Q} -vector space. Also the Chow groups of the incidence sub variety $Z \subset \text{Gr}(3, H^0(\mathcal{O}_{\mathbf{P}^5}(2))) \times \mathbf{P}^5$ are finite dimensional \mathbf{Q} -vector spaces. Therefore $A^2(\mathcal{X})$ and $A^2(\mathcal{X}_U)$, with U a open subset

of S , are finite dimensional \mathbf{Q} -vector spaces, so that, as in the previous cases, we get $A_0(X_\eta) \simeq \mathbf{Q}$. \square

Remark 5.8. If X is a general polarized K3 surface of genus g with $g > 5$ then the projective model of X is not a complete intersection in \mathbf{P}^g . However S.Mukai proved, in [Mu 1] and [Mu 2], that, for $6 \leq g \leq 10$, and also for $g = 12, 13, 18, 20$, X is still a complete intersection with respect to a homogeneous vector bundle in a g -dimensional Grassmanian. So an argument similar to the one used in the proof of the Corollary above may be used also in this cases. The following result takes care of the case $g = 6$

Corollary 5.9. *Let $f : \mathcal{X} \rightarrow S$ be the universal family of polarized K3 surfaces of genus g , with $g = 6$. Then $A_0(X_\eta) \simeq \mathbf{Q}$, with X_η the generic fibre of f .*

Proof. A general polarized K3 surface of genus 6 can be obtained as a complete intersection in the Grassmanian $\mathrm{Gr}(2, 5)$ of 2-dimensional subspaces in a fixed 5-dimensional vector space. $G = \mathrm{Gr}(2, 5)$ is embedded into \mathbf{P}^9 by Plücker coordinates and has dimension 6 and degree 5. The intersection in \mathbf{P}^9 of G with 3 hyperplanes H_1, H_2, H_3 is a Fano 3-fold F_5 of index 2 and degree 5. The isomorphism class of F_5 does not depend on the choice of the 3 hyperplanes. A smooth complete intersection of F_5 with a quadratic hypersurface is a K3 surface of genus g . Let \mathcal{V} be the projective bundle $|3\mathcal{O}_G(1) \oplus \mathcal{O}_G(2)|$ over G and let \mathcal{X} be the incidence variety $Z \subset G \times |3\mathcal{O}_G(1) \oplus \mathcal{O}_G(2)|$, given by couples (x, P) where $x \in G$ belongs to the intersection of the hyperplanes and the quadratic hypersurfaces corresponding to $P \in |3\mathcal{O}_G(1) \oplus \mathcal{O}_G(2)|$. Let $f : \mathcal{X} \rightarrow G$ be the map induced by the first projection. The Chow ring $A(G)$ is a finite dimensional \mathbf{Q} -vector space. Let $U \subset G$ be a Zariski open subset such that the projective bundle \mathcal{V} is trivial over U . Then $A^2(\mathcal{X}_U) \simeq A^2(U \times (\mathcal{V})_U)$ is a finite dimensional \mathbf{Q} -vector space. Therefore $A_0(X_\eta) \simeq \mathbf{Q}$, where X_η is the generic fibre of $f : \mathcal{X} \rightarrow G$. \square

The following lemma appears in [GG,6.1].

Lemma 5.10. *Let $f : \mathcal{X} \rightarrow C$ be a smooth projective family of surfaces over an algebraically closed field of characteristic 0, with $\dim C = 1$. Let η be the generic point of C and let s be a closed point. Let \bar{K} be the algebraic closure of $K = k(\eta)$ and let $X_{\bar{K}} = X_\eta \times_K \bar{K}$, where X_η is the generic fibre of f . Then if $A_0(X_{\bar{K}}) \simeq \mathbf{Q}$ also $A_0(X_s) \simeq \mathbf{Q}$, where X_s is the fibre of f over the closed point s .*

Proof. Let R be the completion of the local ring of C at s . Then, passing to colimits over finite extensions of R , we have a specialization homomorphism over algebraically closed fields of characteristic 0

$\sigma : A^i(X_{\bar{K}}) \rightarrow A^i(X_s)$ and a commutative diagram

$$\begin{array}{ccc} A^i(X_{\bar{K}}) & \xrightarrow{\sigma} & A^i(X_s) \\ cl \downarrow & & cl \downarrow \\ H^{2i}(X_{\bar{K}}, \mathbf{Q}_l(i)) & \xrightarrow{\simeq} & H^{2i}(X_s, \mathbf{Q}_l(i)) \end{array}$$

where H^* is l -adic cohomology, see [Fu, 20.3.5]. Now assume $A_0(X_{\bar{K}}) \simeq \mathbf{Q}$. Then, by the results in [BS], $v(\Delta_{X_{\bar{K}}}) = 0$. Therefore $t_2(X_{\bar{K}}) = 0$, because the identity map on $t_2(X_{\bar{K}})$ is 0 in $\mathcal{M}_{rat}(\bar{K})$. It follows that the motive $h(X_{\bar{K}})$ is finite dimensional. Also the group $H^2(X_{\bar{K}}, \mathbf{Q}_l(1))$ is algebraic. The finite dimensionality of the motive $h(X_{\bar{K}})$ in $\mathcal{M}_{rat}(\bar{K})$ implies, via the specialization map σ , the finite dimensionality of $h(X_s)$ in $\mathcal{M}_{rat}(k)$, see [Ped 1, 4.3]. From the commutative diagram we also get the algebraicity of $H^2(X_s, \mathbf{Q}_l(1))$. Therefore $t_2(X_s) = 0$ which implies $A_0(X_s) \simeq \mathbf{Q}$ \square

Corollary 5.11. *Let $f : \mathcal{X} \rightarrow \mathbf{P}_k^n$ be a smooth projective family of surfaces over an algebraically closed field k of characteristic 0. Let X_η be the generic fibre of f , which is a surface over $K = k(t_1, \dots, t_n)$, and let X_P be the fibre over a closed point $P \in \mathbf{P}_k^n$. Assume that $A_0(X_{\bar{K}}) \simeq \mathbf{Q}$. Then also $A_0(X_P) \simeq \mathbf{Q}$.*

Proof. Let \mathcal{M} be the maximal ideal of $A = k[t_1, \dots, t_n]$, corresponding to the closed point P , and let $A_{\mathcal{M}} \simeq k[t_1, \dots, t_n]_{(t_1, \dots, t_n)}$ be the local ring of \mathbf{P}_k^n at P . Let $S = \text{Spec } A_{\mathcal{M}}$ and let $\tilde{f} : \tilde{\mathcal{X}} \rightarrow S$ be the induced fibration. Let $R_n = A_{\mathcal{M}} / (t_1, \dots, t_{n-1})A_{\mathcal{M}}$. Then R_n is a local ring of dimension 1 with quotient field $k(t_n)$ and residue field k . Let $S_n = \text{Spec } R_n$ and let

$$\begin{array}{ccc} \tilde{\mathcal{X}}_n & \longrightarrow & \tilde{\mathcal{X}} \\ \tilde{f}_n \downarrow & & \tilde{f} \downarrow \\ S_n & \longrightarrow & S \end{array}$$

be the base change. The generic fibre $(\tilde{X}_n)_{\eta_n}$ of \tilde{f}_n is a surface over $k(t_n)$ and the closed fibre the surface X_P over k . Let \bar{K}_n be the algebraic closure of the field $k(t_n)$, Then $\bar{K}_n \subset \bar{K}$ and $A_0(\tilde{X}_n)_{\bar{K}_n} \simeq A_0(\tilde{X}_{\bar{K}}) \simeq \mathbf{Q}$, because the base change from an algebraically closed field to a larger algebraically closed field induces an isomorphism on Chow groups with \mathbf{Q} -coefficients. From lemma we get $A_0(X_P) \simeq \mathbf{Q}$. \square

Remark 5.12. Let $f : \mathcal{X} \rightarrow S$ be one of the families of polarized K3 surfaces of genus g considered in Corollary 5.7 and in Corollary 5.9, where S is either isomorphic or birational to $\mathbf{P}_{\mathbf{C}}^n$, for some n . Let X_η be the generic fibre of f and let K be the algebraic closure of $k(\eta) = k(S)$.

Then, from Corollary 5.11, we get $A_0(X_K) \neq \mathbf{Q}$, because $A_0(X_s) \neq \mathbf{Q}$ for a closed fibre X_s , which is a K3 surface over \mathbf{C} .

Proposition 5.13. *Let $f : \mathcal{X} \rightarrow C$, with $\dim C = 1$, be a smooth projective family of K3 surfaces over an algebraically closed field k . Then the following conditions are equivalent:*

- (i) $A_0(X_\eta) \simeq \mathbf{Q}$, where η is the generic point of C ;
- (ii) $A^2(\mathcal{X})$ is a finitely dimensional \mathbf{Q} -vector space.

Proof. (i) \implies (ii). For every open subset $U = C - \{P_1, \dots, P_n\}$ the localization sequence in (5.4) gives

$$(5.14) \quad \bigoplus_{1 \leq i \leq n} A^1(X_{P_i}) \rightarrow A^2(\mathcal{X}) \rightarrow A^2(\mathcal{X}_U) \rightarrow 0$$

where the groups $A^1(X_{P_i}) = NS(X_{P_i})\mathbf{Q}$ are finitely generated vector spaces. The category of Chow motives over the generic point η of C equals the colimit of categories of Chow motives over non-empty open subsets U in the base curve C , i.e there is a functor F

$$F : \mathcal{M}_{rat}(\eta) \xrightarrow{\simeq} \operatorname{colim}_{U \subset C} \mathcal{M}_{rat}(U)$$

which is an equivalence of categories, see [Gu, Lemma 4]. The functor F is obtained by taking spreads of algebraic cycles and localizations sequences for Chow groups. If $M = (X, p)$ is a Chow motive over $k(\eta)$ then $F(M) = (Y', p')$, where Y' and p' are spreads of Y and p over some open subset $U \subset C$. Also, by eventually shrinking U , we may assume that p' is a projector, so that $M' = (Y', p')$ is a Chow motive over $k(U)$. On morphisms F is defined in a similar way, because morphisms in a category of Chow motives are correspondences. We have

$$\begin{aligned} A^2(X_\eta) &= \operatorname{Hom}_{\mathcal{M}_{rat}(k(\eta))}(\mathbf{1}, h(X_\eta)) = A_0(\operatorname{Spec} k(\eta) \times X_\eta) \simeq \\ &\simeq \mathbf{Q} \simeq \operatorname{Hom}_{(\mathcal{M}_{rat}(k(\eta))(\operatorname{Spec} k(\eta) \times \operatorname{Spec} k(\eta)))}. \end{aligned}$$

Therefore there exists an open subset $U \subset C$ such that $A^2(\mathcal{X}_U) = \operatorname{Hom}_{\mathcal{M}_{rat}(k(U))}(\operatorname{Spec} k(U) \times \operatorname{Spec} k(U))$, i.e $A^2(\mathcal{X}_U) \simeq \mathbf{Q}$. From the localization sequence we get that $A^2(\mathcal{X})$ is a finite dimensional \mathbf{Q} -vector space.

(ii) \implies (i). From the exact sequence in (5.14) $A^2(\mathcal{X}_U)$ is a finite dimensional \mathbf{Q} -vector space, for every open $U \subset C$. From (5.5) we get $\lim_{U \subset C} A^2(\mathcal{X}_U) = A^2(X_\eta)$ and hence $A_0(X_\eta) \simeq \mathbf{Q}$. \square

Remark 5.15. Let $f : \mathcal{X} \rightarrow C$ be a smooth projective family of surfaces on $k = \bar{k}$ with C a smooth curve. In [GG] it has been proved that, if $A_0(\mathcal{X}) \simeq \mathbf{Q}$, then $A^2(\mathcal{X})$ is a finite dimensional \mathbf{Q} -vector space and \mathcal{X} has a finite dimensional motive, which lies in the subcategory

of $\mathcal{M}_{rat}(k)$ generated by the motives of abelian varieties. This is the case if \mathcal{X} is a Fano 3-fold.

Example 5.16. (Huybrechts) Let $\mathcal{X} \rightarrow \mathbf{P}^1$ be a general Lefschetz pencil of quartics in \mathbf{P}_k^3 , with k an algebraically closed field of characteristic 0, such that $\mathcal{X} = Bl_C(\mathbf{P}^3)$ where C is a complete intersection curve of genus $g > 0$ and $A^2(\mathcal{X}) \simeq \mathbf{Q} \oplus \text{Pic } C$. Let X_η be the generic fibre. Then $A_0(X_\eta) \neq \mathbf{Q}$ because $A^2(\mathcal{X})$ is not a finite dimensional \mathbf{Q} -vector space.

REFERENCES

- [AC] E.Arbarello and M.Cornalba, *The Picard group of the moduli space of curves*, Topology **26**(1987), 153-171
- [BV] A. Beauville and C. Voisin *On the Chow ring of a K3 surface*, J.Alg geom. **13**,(2004),417-426
- [BS] S. Bloch and V.Srinivas *Remarks on correspondences and algebraic cycles* , American J.of Math **105** (1983) 1234-1253
- [Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, Heidelberg-New-York, 1984.
- [GS] A. Garbagnati and A.Sarti, *On symplectic and non-symplectic automorphisms of K3 surfaces* Revista Matematica Iberoamericana **29** (2013), no. 1.
- [GG] S.Gorchinskiy and V.Guletskii *Motives and representability of algebraic cycles on threefolds over a field*, J.Alg. Geometry **21** (2012) 343-373.
- [Gu] V.Guletskii *On the continuous part of codimension 2 algebraic cycles on three-dimensional varieties*, Sbornik Math. **200**:3,(2009),325-338.
- [Huy 1] D.Huybrechts, *Symplectic automorphisms of K3 surfaces of arbitrary order*, Math.Res.Lett. **19** (2012) 947-951
- [Huy 2] D.Huybrechts, (with an appendix by C. Voisin) *Curves and cycles on K3 surfaces* Algebraic Geometry **1** (2014), 69-106. arXiv:1303.4564
- [Jan] U.Jannsen, *Motivic sheaves and Filtrations on Chow groups*, Proceedings of Symposia in Pure Mathematics, vol.55,Part 1 (1994),245-302
- [KMP] B. Kahn, J. Murre and C. Pedrini, *On the transcendental part of the motive of a surface*, pp. 143–202 in "Algebraic cycles and Motives Vol II", London Math. Soc. LNS **344**, Cambridge University Press, 2008.
- [Ki] S.I.Kimura *Chow groups are finite dimensional, in some sense*, Math. Ann. **331** no. 1 (2005), 173-201
- [Muk 1] S.Mukai *Curves, K3 surfaces and Fano's 3-folds of genus ≤ 10* , Algebraic Geometry and Commutative algebra in Honor of M.Nagata, Academic Press (1987) 355-377.
- [Muk 2] S.Mukai *Polarized K3 surfaces of genus thirteen*, Advances Studies in Pure Math. **45**,(2006) 315-326
- [MNP] J.Murre, J.Nagel and C.Peters, *Lectures on the theory of pure Motives*, AMS University Lectures Ser. Vol 61 (2013)
- [Ni] V. Nikulin *Finite automorphisms of Kählerian surfaces of type K3*, Trans. Moscow Math. Society, **38** (1980),71-137.
- [O'Gr] K.G.OGrady, *Moduli of sheaves and the Chow group of K3 surfaces*, J. Math. Pures Appl. **100** (2013), 701718.

- [Ped 1] C.Pedrini *On the motive of a K3 surface*, in "The geometry of algebraic cycles", Clay Math. Proceedings , Vol **9**, (2010), 53-73
- [Ped 2] C.Pedrini *On the finite dimensionality of a K3 surface*, Manuscripta Math. **138**, (2012), 59-72
- [PW 1] C.Pedrini and C.Weibel *Severi's results on correspondences*, Rend. Sem. Mat. Univ. Politec. Torino Vol. 71, 3-4 (2013),493-504
- [PW 2] C.Pedrini and C.Weibel *Some surfaces of general type for which Bloch's conjecture holds* to appear on "Proceeding of the Conference on "Period Domains, Algebraic Cycles, and Arithmetic", Vancouver ,2013 Cambridge University Press
- [Sev] F. Severi, *La teoria delle corrispondenze a valenza sopra una superficie algebrica: il principio di corrispondenza Nota III*, Rend. Reale Acc. Naz. Lincei, Classe di Scienze, Vol XVII (1933), 869-881.
- [VG-S] B.Van Geemen and A. Sarti, *Nikulin involutions on K3 surfaces*, Math.Z. **255**, (2007),731-753
- [Vois 1] C.Voisin, *Symplectic involutions on K3 surfaces act trivially on CH_0* , Documenta Math. 17 (2012) 851-860.
- [Vois 2] C.Voisin, *Bloch's conjecture for Catanese and Barlow surfaces* J. Differential Geometry 97 (2014) 149-175.
- [Vois 3] C.Voisin, *Rational equivalence of 0-cycles on K3 surfaces and conjectures of Huybrechts and O'Grady* preprint 2012, to appear in "Recent Advances in Algebraic Geometry, a conference in honor of Rob Lazarsfeld's 60th birthday".
- [Vois 4] C.Voisin *Chow groups, decomposition of the diagonal and the topology of families*, Annals of Math. Studies 187, Princeton University Press 2014.